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THESIS

SOME STOCHASTIC-DUEL MODELS OF COMBAT

by

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Some Stochastic-Duel Models of Combat

by

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This paper provides the conceptual foundation for stochastic-duels and then develops a modest extension to more realistic combat situations. Simple stochastic models for the fundamental duel and the classical duel are reviewed. A modest extension is developed for the theory of multiple duels: when all firing times are continuous random variables, an expression for the probability of winning such a duel is derived by using the theory of continuous-time Markov chains.

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I. INTRODUCTION

In the nineteenth century, Von Clausewitz [Ref. 5] remarked that "war is nothing but a duel on a large scale." Subsequently, in the twentieth century, the theory of stochastic-duels was developed by C. J. Ancker [Refs. 2, 3, and 4] and others to mathematically look at such duels in order to have a mathematical basis for studying modern combat. Thus, the theory of stochastic duels considers combat at a microscopic level (individual fires opposing each other), whereas at the other extreme the Lanchester theory of warfare considers it at a macroscopic level (large groups of homogeneous fires opposing each other). This thesis will review the conceptual foundation of the theory of stochastic duels (in particular, one-on-one duels) and then develop a modest extension to more realistic combat situation (namely, two-on-one duels).

Additionally, the author hopes that his exposition about this material concerning one-on-one duels makes the concept more accessible to the professional military officers. Thus this expository material strives to be simple (but yet complete) and self-contained (and hence full details will be supplied to the reader). It also sets the stage for the extension to multiple fires (i.e., the two-on-one duel).

Let us now consider the nature of the theory of stochastic duels in more detail. It is concerned with the microscopic features of combat such as kill probabilities of individual rounds, times between rounds fired, ammunition limitations, etc. In the theory of stochastic duels, two duellists (usually denoted as A and B) fire at each other until one

or the other has been killed. The times between the firing of successive rounds by each duellist are frequently taken to be random variables, pairwise independent. The simplest case is that in which there is a single duellist on each side (i.e., one-on-one duel).

There are two basic cases for stochastic duels that have been distinguished in the literature: 1) the fundamental duel, and 2) the classical duel. In the fundamental duel, the two duellists have unlimited ammunition and each starts with an unloaded weapon. Specific solutions have been derived for a general firing-time distribution and also for exponentially-distributed firing times. Later in this thesis we will give a simple development of the exponential firing time results. In the classical duel, each duellist starts with a loaded weapon, they fire simultaneously at the beginning of the duel, and then they proceed as in the fundamental duel. When the firing time is discrete, the solution for the stochastic duel has been derived by using a special technique [Ref. 3]. When the firing time is continuous, the solution for the stochastic duel is derived by using the theory of continuous-time Markov chains. In Chapter IV, a numerical example is considered and corresponding parametric results are graphically presented.

II. SOME BASIC STOCHASTIC-DUEL MODELS

In this chapter we will consider some simple (but yet basis) stochastic-duel models for: 1) the fundamental duel, and 2) the classical duel. In the fundamental duel, the duellists each start with an unloaded weapon, load their weapons, and then fire at each other until one of them is finally killed. In the classical duel, they both start with loaded weapons, fire their first rounds simultaneously, and then proceed as in the fundamental duel. In this chapter, specific solutions are derived for both the fundamental duel and also the classical duel for the special case of exponential firing times (which is of fundamental importance for understanding future enhancements).

A. THE FUNDAMENTAL DUEL

In the fundamental duel, two duellists, A and B, start with unloaded weapons and then fire at each other until one is killed. A's firing time (the time between rounds) is a random variable with a known probability density, $f_A(t)$. B's firing time is similarly characterized by the density, $f_B(t)$. Successive firing times are selected from $f_A(t)$ and $f_B(t)$, independently and at random. Each time A fires, he has a fixed probability p_A of killing B. We will denote the probability that B is not killed as q_A, and hence $p_A + q_A = 1$. Similarly denoted as p_B, with its complement being similarly defined (i.e., $p_B + q_B = 1$). After the starting signal, each contestant loads his weapon, aims, and then fires his first round. In other words, in the fundamental duel the duellists

start with unloaded weapons. Both (A and B) have unlimited supplies of ammunition that, among other things, makes a kill by one of them an ultimate certainty. A wins if he is the one to first score a kill. The probability of this will be denoted as P(A), and p(A) + p(B) = 1, where p(B) denotes the probability that B wins.

1. Development of Results for Fundamental-Duel Model

In this section we develop an expression for the probability that Combatant A wins a "fundamental duel" against Combatant B, denoted as p(A), in the case in which the firing times are exponentially distributed. Our final results for p(A) is given by equation (15) below.

In order to develop an expression for the probability that A wins the duel, we consider the combatants to be decoupled, i.e., each combatant fires at a passive target (one that does not return fire). Let $k_A(t)$ denote the probability density for the time for A to kill his passive target and $K_A(t)$ denote the corresponding cumulative distribution function, i.e.,

$$K_A(t) = \int_0^t k_A(s)\, ds$$

We similarly define $k_B(t)$ and $K_B(t)$, i.e.

$$K_B(t) = \int_0^t k_B(s)\, ds$$

Then in order for A to win the duel he must kill his target before B kills B's target. In other words

$$P(A) = \text{Prob}\,[T_A < T_B], \tag{1}$$

Where T_A denotes the time [the random variable corresponding to $k_A(t)$] and similarly for T_B [Ref. 6].

$$p(A) = \int_0^t \{1 - K_A(s)\}\, d\, K_B(s) \tag{2}$$

or

$$p(A) = \int_0^t \{1 - K_A(s)\}\, d\, k_B(s)\, ds \tag{2}$$

The above expression holds in general, but we still must develop expression $k_A(t)$ and $k_B(t)$ based on our model. In other words, if we assume that, for example, we know the distributions of firing times and know the corresponding single-shot kill probabilities, we must combine these into a time-to-kill distribution.

Thus, we assume that A's firing time (i.e., the times between rounds) are exponentially and identically distributed, with common probability density as $f_A(t)$. Thus

$$f_A(t) = r_A e^{-r_A t}$$

where r_A denotes the firing rate of A. If we assume that the probability that A kills his target with any one round is consistant for all rounds and denote this probability as p_A, then

$$\text{Prob}\left[\begin{matrix}\text{nth round kills}\\ \text{target}\end{matrix}\right] = p_A q_A^{\,n-1} \tag{3}$$

where $q_A = 1-p_A$. Thus,

$$\text{Prob}\left[\begin{matrix}\text{A takes time between t}\\ \text{and t+}\Delta\text{t to kill target}\end{matrix}\right] = \sum_{n=1}^{\infty} \text{Prob}\left[\begin{matrix}\text{nth rounds}\\ \text{kills target}\end{matrix}\right]$$

$$\cdot\ \text{Prob}\left[\begin{matrix}\text{A fires nth rounds}\\ \text{between t and t+}\Delta\text{t}\end{matrix}\right] \tag{4}$$

now

$$\text{Prob}\left[\begin{matrix}\text{A fires nth rounds}\\ \text{between t and t+}\Delta\text{t}\end{matrix}\right] = \text{Prob}\left[\begin{matrix}\text{A has fired}\\ \text{(n-1) rounds by t}\end{matrix}\right]$$

$$\cdot\ \text{Prob}\left[\begin{matrix}\text{A fires one more}\\ \text{round from t to t+}\Delta\text{t}\end{matrix}\right]$$

then

$$\text{Prob}\left[\begin{matrix}\text{A fires nth rounds}\\ \text{between t and t+}\Delta\text{t}\end{matrix}\right] = \frac{(r_A t)^{n-1}}{(n-1)!} e^{-r_A t} \cdot r_A \Delta t \tag{5}$$

or

$$\text{Prob}\left[\begin{matrix}\text{A fires nth rounds}\\ \text{between t and t+}\Delta\text{t}\end{matrix}\right] = \frac{r_A^{\,n} t^{n-1}}{(n-1)!} e^{-r_A t}\, \Delta t \tag{6}$$

Since [Ref. 1]

$$\text{Prob}\left[\begin{array}{c}\text{A has fired}\\ \text{(n-1) rounds by t}\end{array}\right] = \frac{(r_A t)^{n-1}}{(n-1)!}\, e^{-r_A t} \tag{7}$$

and

$$\text{Prob}\left[\begin{array}{c}\text{A fires one round}\\ \text{between t and t+}\Delta\text{t}\end{array}\right] = r_A \Delta t \tag{8}$$

Substituting (3) and (6) into (4), we obtain

$$\text{Prob}\left[\begin{array}{c}\text{A takes time between}\\ \text{t and t+}\Delta\text{t to kill target}\end{array}\right] = \sum_{n-1}^{\infty} p_A q_A^{\,n-1} \cdot \frac{r_A{}^{n} t^{n-1}}{(n-1)!}\, e^{-r_A t} \Delta t$$

$$= r_A p_A\, e^{-r_A t} \cdot \Delta t \sum_{n=1}^{\infty} \frac{(q_A r_A \cdot t)^{n-1}}{(n-1)!} \tag{9}$$

or

$$\text{Prob}\left[\begin{array}{c}\text{A takes time between t}\\ \text{and t+}\Delta\text{t to kill target}\end{array}\right] = p_A r_A\, e^{-p_A r_A t} \tag{10}$$

Thus

$$k_A(t) = p_A r_A \, e^{-p_A r_A \cdot t} \tag{11}$$

and

$$K_A(t) = e^{-p_A r_A \cdot t} \tag{12}$$

Similarly,

$$k_B(t) = p_B r_B \, e^{-p_B r_B \cdot t} \tag{13}$$

and

$$K_B(t) = e^{-p_B r_B \cdot t} \tag{14}$$

Substituting (12) and (13) into (2), we find that

$$P(A) = \frac{p_A r_A}{p_A r_A + p_B r_B} \tag{15}$$

which is our final result.

B. THE CLASSICAL DUEL

In contrast to the fundamental duel, two duellists, A and B, start with loaded weapons, fire their first rounds simultaneously, and then proceed as in the fundamental duel. In order to develop an expression for the probability that A wins a "classical duel" against Contestant B, denoted as P(A), in the case in which the firing time are exponentially distributed. The final solution p(A) is given by equation (21) below.

$$\text{Prob [A wins]} = \left[\begin{array}{l} \text{A kills B on} \\ \text{the 1st round} \end{array} \right] \cdot \left[\begin{array}{l} \text{B does not kill A} \\ \text{on the 1st round} \end{array} \right]$$

$$+ \text{ Prob} \left[\begin{array}{l} \text{Neither is killed} \\ \text{on the 1st round} \end{array} \right] \cdot \text{Prob} \left[\begin{array}{l} \text{A wins the} \\ \text{subsequent duel} \end{array} \right] \qquad (16)$$

now

$$\text{Prob} \left[\begin{array}{l} \text{A kills B on} \\ \text{the 1st round} \end{array} \right] = p_A \qquad (17)$$

$$\text{Prob} \left[\begin{array}{l} \text{B does not Kill A} \\ \text{on the 1st round} \end{array} \right] = q_B \qquad (18)$$

$$\text{Prob} \left[\begin{array}{l} \text{Neither is killed} \\ \text{on the 1st round} \end{array} \right] = q_A \cdot q_B \qquad (19)$$

$$\text{Prob} \left[\begin{array}{l} \text{A wins the} \\ \text{subsequent duel} \end{array} \right] = P(A)_f = \frac{p_A r_A}{p_A r_A + p_B r_B} \qquad (20)$$

where $P(A)_f$: the result of the fundamental duel substituting (10), (18), (19), and (20) into (16), we find

$$P(A) = \frac{p_A q_B (p_B r_B + r_A)}{p_A r_A + p_B r_B} \tag{21}$$

which is our final result. But in the classical duel, the following case will happen, i.e., Contestant A and Contestant B will be killed on the first round. Therefore

$$P(A) + P(B) \neq 1$$

thus

$$P(A) + P(B) + P(AB) = 1$$

where p(AB): the probability that both are killed on the first round.

$$P(AB) = 1 - P(A) - P(B) = p_A p_B \tag{22}$$

III. AN EXTENSION TO MULTIPLE FIRES

A. DISCRETE FIRING TIME

In a discrete firing time, two duellists, A and B, start with unlimited ammunition, fire at each other with fixed kill probabilities p_A of killing B. Similarly denoted as p_B of killing A. They start with unloaded weapons and fire at fixed intervals a and b respectively. This is similar to a situation in which each duellist is armed with an automatic weapon.

1. Development of Results for Fundamental-Duel Model

In order to develop an expression for the probability that A wins the fundamental-duel, we will assume that a and b (fixed firing interval) are rational numbers if a and b can be reduced to α/β where α and β are relatively prime integers. And we define

$$\frac{\alpha}{\beta} = n \;\ldots\ldots r \qquad \alpha = n\beta + r \tag{23}$$

where n is an integer and r is the remainder.

The total probability of A's total success on the jth rounds [Ref. 3], i.e.

$$P\left[\begin{matrix}\text{A's total success}\\ \text{on the jth round}\end{matrix}\right] = \sum_{j=1}^{j=\infty} P\left[\begin{matrix}\text{first j-1th}\\ \text{round fail}\end{matrix}\right] \cdot P\left[\begin{matrix}\text{Kill on the}\\ \text{jth rounds}\end{matrix}\right]$$

$$\cdot\, P\left[\begin{matrix}\text{B is falling on}\\ \text{his first K round}\end{matrix}\right] \tag{24}$$

where $$K = j\,\frac{\alpha}{\beta}\ .$$

then

$$P\left[\begin{array}{l}\text{A's total success}\\ \text{on the jth round}\end{array}\right] = \sum_{j=1}^{\infty} (q_A)^{j-1}\,(p_A)\,(q_B)^{k} \tag{25}$$

or

$$P\left[\begin{array}{l}\text{A's total success}\\ \text{on the jth round}\end{array}\right] = p_A q_B^{\ n} \sum_{j=0}^{\infty} q_A^{\ j} q_B^{\ jn+[(j+1)(\frac{r}{\beta})]} \tag{26}$$

let

$$(j+1)\,(\tfrac{r}{\beta}) = [x_j]$$

where $[x_j]$: largest integer equal to or less than the number x_j

Assume

$$[x_j + k\beta] = [x_j + K_r] = [x_j] + K_\gamma \tag{27}$$

thus,

$$P\left[\begin{array}{l}\text{A's total success}\\ \text{on the jth round}\end{array}\right] = \left\{\frac{p_A q_B^{\ n}}{1-q_A^{\ \beta}\ q_B^{\ \alpha}}\right\} \sum_{j=1}^{\beta=1} q_A^{\ j} q_B^{\ jn+[x_j]}$$

$$= \left\{\frac{p_A}{(1-q_A{}^{\beta}q_B{}^{\alpha})}\right\} \sum_{j=0}^{\beta-1} q_A{}^{j}\, q_B{}^{[(j+1)(\frac{\alpha}{\beta})]}$$

$$= \left\{\frac{p_A\, q_B{}^{n}}{(1-q_A{}^{\beta}q_B{}^{\alpha})}\right\} 1 + q_A\, q_B{}^{n+[x_1]} + q_A{}^{2}\, q_B{}^{2n+[x_2]} + q_A{}^{\beta-1}\, q_B{}^{\alpha-n}$$

$$= \left\{\frac{p_A}{(1-q_A{}^{\beta}q_B{}^{\alpha})}\right\} q_B{}^{[\frac{\alpha}{\beta}]} + q_A q_B{}^{[2\frac{\alpha}{\beta}]} + \ldots + q_A{}^{\beta-1} \cdot q_B{}^{\alpha} \quad \text{[Ref. 3]} \tag{28}$$

where $n = [\frac{\alpha}{\beta}]$, $r = \alpha - n\beta$, and $[x_j] = [(j+1)\,\frac{r}{\beta}]$

Similarly

$$P\left[\begin{matrix}\text{B's total success} \\ \text{on the jth round}\end{matrix}\right] = \left\{\frac{p_B}{(1-q_A{}^{\beta}q_B{}^{\alpha})}\right\} \sum_{K=0}^{\alpha-1} q_B{}^{K} \cdot q_A{}^{[(K+1)\frac{\beta}{\alpha}]} . \tag{29}$$

which is our final results for the fundamental duel as the equation (28).

2. Development of Results for Multiple-Duels Model

In this section we develop an expression for the probability that Contestant A wins "multiple-duels" against Contestant B. In this duel, there are two contestants on the A's side and one contestant on the B side as shown in Figure 1.

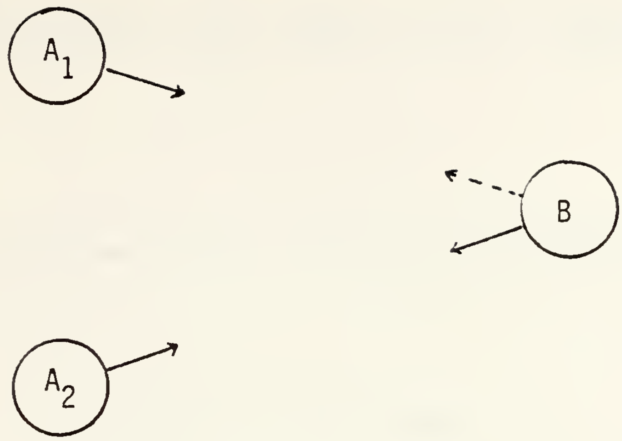

Figure 1. The Situations of Duel

Each time A (A_1, A_2) fires, A has a fixed probability p_A of killing B. We will denote the probability that B is not killed as q_A, and hence $p_A + q_A = 1$. Similarly denoted as p_B, with its complement being similarly defined (i.e., $p_B + q_B = 1$). Both (A and B) have unlimited ammunitions. If the B contestant kills an A_1 (or A_2) he immediately shifts his fire to the remaining A. In this situation, the probability that the side "A" can win is the following:

$$P \left[\begin{array}{c} \text{The side} \\ \text{"A" wins} \end{array} \right] = \left[\begin{array}{c} \text{"A" side kills B and} \\ \text{both } A_1 \text{ and } A_2 \text{ survive} \end{array} \right]$$

$$+ \left[\begin{array}{c} \text{"A" side kills B and one "A" } (A_1 \text{ or } A_2) \\ \text{are to be killed and only one A survivor} \end{array} \right]$$

thus

$$P\left[\begin{matrix}\text{Both } A_1 \text{ and} \\ A_2 \text{ survive}\end{matrix}\right] = P\{A_1 \text{ or } A_2 \text{ or both kill B}\} \cdot p\{\text{B fails to kill}\}$$

$$= \sum_{j=1}^{\infty} p\,\{\text{on j-1 rounds no kills}\} \cdot p\,\{A_1 \text{ or } A_2 \text{ or both kill B on jth round}\}$$

$$\cdot\, p\,\{\text{B fail to jth round}\}$$

$$= \sum_{j=1}^{\infty} (q_A^{\;2} \cdot q_B)^{j-1} \cdot (1 - q_A^{\;2}) \cdot q_B \;=\; \frac{q_B\,(1 - q_A^{\;2})}{(1 - q_A^{\;2} \cdot q_B)} \tag{30}$$

and

$$P\,[\text{ one A } (A_1 \text{ or } A_2) \text{ survive }] = \sum_{j=1}^{\infty} p\,(\text{no kill on j-1 round})$$

$$\cdot\, \{p\,(\text{B kill } A_1 \text{ or } A_2 \text{ and A fail to B})\; P_f(A)$$

$$+\, p\,(\text{B kill one A and A kill B})\}$$

thus

$$P\left[\begin{matrix}\text{one A } (A_1 \text{ or} \\ A_2) \text{ survive}\end{matrix}\right] = \sum_{j=1}^{\infty} (q_A{}^2 \cdot q_B)^{j-1} \cdot P_B \cdot q_A{}^2 \quad P_f(A)$$

$$+ \sum_{j=1}^{\infty} (q_A{}^2 \cdot q_B)^{j-1} \cdot p_B \, (1 - q_A{}^2) \tag{31}$$

where $P_f(A)$ is the results of a fundamental duel in which a=b (fixed firing time).

Thus,

$$P_f(A) = \frac{p_A \, q_B}{(1-q_A \cdot q_B)} \quad \text{[from the equation (28)]} \tag{32}$$

Substituting equation (32) into equation (31), we find that:

$$P\left[\begin{matrix}\text{The side} \\ \text{"A" wins}\end{matrix}\right] = P\left[\begin{matrix}\text{Both } A_1 \text{ and} \\ A_2 \text{ survive}\end{matrix}\right] + P\left[\begin{matrix}\text{One A } (A_1 \text{ or} \\ A_2) \text{ survive}\end{matrix}\right]$$

$$= \frac{p_A \, (1 + q_A \, p_B - q_A{}^2 \cdot q_B{}^2)}{(1 - q_A \, q_B) \, (1 - q_A{}^2 \cdot q_B)} \tag{32}$$

Similarly,

$$P\left[\begin{matrix}\text{The side}\\ \text{"B" wins}\end{matrix}\right] = \sum_{j=1}^{\infty} (q_A{}^2 \cdot q_B)^{j-1} \cdot p_B \cdot q_A{}^2 \cdot P_f(B) \qquad (33)$$

where $P_f(B)$ is the results of the fundamental-duel in which a=b.

Therefore,

$$P\left[\begin{matrix}\text{The side}\\ \text{"B" win}\end{matrix}\right] = \frac{p_B{}^2 \cdot q_A{}^3}{(1 - q_A\, q_B)\,(1 - q_A{}^2 \cdot q_B)} \qquad (34)$$

Let us denote P(AB) the probability of draw.

Then,

$$P(AB) = \sum_{j=1}^{\infty} p\ (\text{no kills on j-1 round}) \cdot p\ (\text{B kill one A})$$

$$\cdot\ p\ (\text{A does not kill B}) \cdot p\left(\begin{matrix}\text{one A and B have}\\ \text{duel of draw}\end{matrix}\right)$$

$$= \sum_{j=1}^{\infty} (q_A{}^2 \cdot q_B)^{j-1} \cdot (P_B) \cdot (q_A{}^2) \cdot P_f(AB) \qquad (35)$$

where $P_f(AB)$ is the result of the fundamental duels with a=b.

$$P_f(AB) = \frac{p_A \, p_B \, q_A^{\beta-1} \, q_B^{\alpha-1}}{1 - q_A^{\beta} \, q_B^{\alpha}} \tag{36}$$

But when a=b,

$$P_f(AB) = \frac{p_A \, p_B}{1 - q_A \, q_B} \tag{37}$$

Substituting equation (37) into equation (35)

$$P(AB) = \frac{p_A \, q_A^2 \, p_B^2}{(1 - q_A q_B)\,(1 - q_A^2 \cdot q_B)} \tag{38}$$

which is our final solution as the equation (32) and equation (34).

B. CONTINUOUS FIRING TIME

In this duel, two duellists, A and B, start with unloaded weapons and then fire at random. But B's sides has two weapon systems and A's sides has only one weapon system. A's firing time is a random variable with a known probability density, $f_A(t)$. B's firing time is similarly characterized by the density, $f_B(t)$. Successive firing times are selected from each density independently. We will denote r the time between round fired (i.e., r_A for A system and r_B for B systems) and the

firing interval between rounds is independent. Both systems has unlimited ammunition and fire each other with fixed kill probability p_A for A system and p_B for B system as shown in Figure 2.

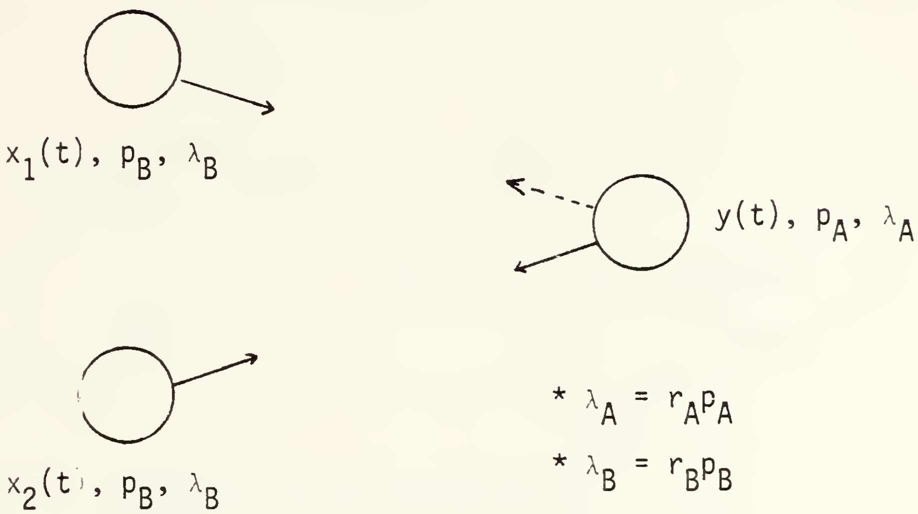

Figure 2. Combat Situations

If we assume that y(t) and x(t) are the state of each weapon system at time t, then

$$y(t) = \begin{cases} 1 : \text{A contestant was not killed} \\ 0 : \text{A contestant killed} \end{cases}$$

and

$$\begin{matrix} x_1(t) \\ \text{or} \\ x_2(t) \end{matrix} = \begin{cases} 1 : \text{B } (B_1 \text{ or } B_2) \text{ contestant was not killed} \\ 0 : \text{B } (B_1 \text{ or } B_2) \text{ killed.} \end{cases}$$

Let us consider the state of duel in Figure 3.

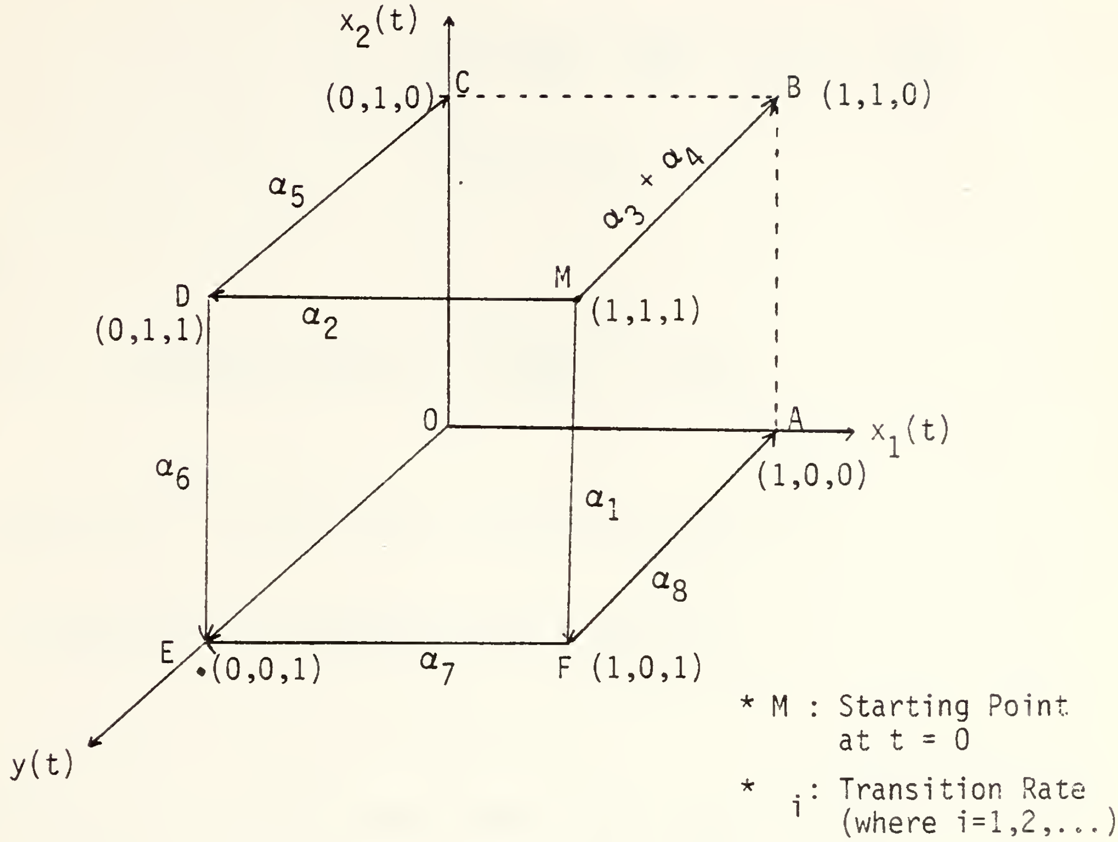

Figure 3. The State of Duel

where points (A), (B), and (C) are the point of B's winning and only point (E) is the point of A's winning. During the Δt, the transition rates are the following:

(1) P [y hit x_2, x_1 miss y and x_2 miss y]

$$= (\tfrac{1}{2}\lambda_A \Delta t) \cdot (1-\lambda_B\Delta t) \cdot (1-\lambda_B\Delta t)$$

$$= \tfrac{1}{2}\lambda_A\Delta t - \lambda_B\Delta t^2 + \tfrac{1}{2}\lambda_A \cdot \lambda_B{}^2\Delta t^3$$

$$= \tfrac{1}{2}\lambda_A \cdot \Delta t$$

therefore

$$\text{Transition rate } \alpha_1 = \frac{\tfrac{1}{2}\lambda_A \cdot \Delta t}{\Delta t} = \tfrac{1}{2}\lambda_A$$

(2) P [y hit x_1, x_2 miss y and x_1 miss y] $= \tfrac{1}{2}\lambda_A \cdot \Delta t$

Similarly, Transition rate $\alpha_2 = \tfrac{1}{2}\lambda_A$.

(3) P [x_1 hit y, x_2 miss y and y miss x_1] $= (\lambda_B\Delta t)\,(1-\lambda_B\Delta t)\,(1-\lambda_A\Delta t)$

$$= \lambda_B\Delta t$$

Transition rate $\alpha_3 = \lambda_B$

(4) P [x_2 hit y, x_1 miss y and y miss x_2] $= \lambda_B\Delta t$

Transition rate $\alpha_4 = \lambda_B$

(5) P [x_2 hit y, and y mis x_2] = $(\lambda_B \Delta t) \cdot (1-\lambda_A \Delta t)$

Transition rate $\alpha_5 = \lambda_B$

(6) P [y hit x_2 and x_2 miss y] = $(\frac{1}{2}\lambda_A \Delta t)\ (1-\lambda_B \Delta t)$

Transition rate $\alpha_6 = \frac{1}{2}\lambda_A$

(7) P [y hit x_1 and x_1 miss y] = $(\frac{1}{2}\lambda_A \Delta t)\ (1-\lambda_B \Delta t)$

Transition rate $\alpha_7 = \frac{1}{2}\lambda_A$

(8) P [x_1 hit y and y miss x_1] = $(\lambda_B \Delta t)\ (1-\lambda_A \Gamma t)$

Transition rate $\alpha_8 = \lambda_B$

If we assume that Pi (i = 1, 2, ... 8) are the transition probability, P(A) and P(B) are the following:

$$P(A) = P_2 \cdot P_6 + P_1\ P_7 \tag{39}$$

and

$$P(B) = P_3 + P_2 \cdot P_5 + P_1 \cdot P_8 \tag{40}$$

Where $$P_1 = \frac{\alpha_1}{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)} = \frac{\frac{1}{2}\lambda_A}{\frac{1}{2}\lambda_A + \frac{1}{2}\lambda_A + \lambda_B + \lambda_B}$$

$$P_2 = \frac{\alpha_2}{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)} = \frac{\frac{1}{2}\lambda_A}{\frac{1}{2}\lambda_A + \frac{1}{2}\lambda_A + \lambda_B + \lambda_B}$$

$$P_2 = \frac{\alpha_3 + \alpha_4}{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)} = \frac{\lambda_A + \lambda_B}{\frac{1}{2}\lambda_A + \frac{1}{2}\lambda_A + \lambda_B + \lambda_B}$$

therefore $$P_1 + P_2 + P_3 = 1 \quad .$$

$$P_5 = \frac{\alpha_5}{(\alpha_5 + \alpha_6)} = \frac{\lambda_B}{\lambda_B + \frac{1}{2}\lambda_A}$$

$$P_6 = \frac{\alpha_6}{(\alpha_5 + \alpha_6)} = \frac{\frac{1}{2}\lambda_A}{\lambda_B + \frac{1}{2}\lambda_A}$$

$$P_7 = \frac{\alpha_7}{(\alpha_7 + \alpha_8)} = \frac{\frac{1}{2}\lambda_A}{\frac{1}{2}\lambda_A + \lambda_B}$$

$$P_8 = \frac{\alpha_8}{(\alpha_7 + \alpha_8)} = \frac{\lambda_B}{\frac{1}{2}\lambda_A + \lambda_B}$$

therefore

$$P(A) = P_2 \cdot P_6 + P_1 \cdot P_7 = \left(\frac{\frac{1}{2}\lambda_A}{\lambda_A + 2\lambda_B}\right)\left(\frac{\frac{1}{2}\lambda_A}{\lambda_B + \frac{1}{2}\lambda_A}\right) + \left(\frac{\frac{1}{2}\lambda_A}{\lambda_A + 2\lambda_B}\right)\left(\frac{\frac{1}{2}\lambda_A}{\frac{1}{2}\lambda_A + \lambda_B}\right) \qquad (41)$$

Similarly

$$P(B) = P_3 + P_2 \cdot P_5 + P_1 \cdot P_8 = 1 - P(A)$$

$$= \left(\frac{2\lambda_B}{\lambda_A + 2\lambda_B}\right) + \left(\frac{\frac{1}{2}\lambda_A}{\lambda_A + 2\lambda_B}\right)\left(\frac{\lambda_B}{\lambda_B + \frac{1}{2}\lambda_A}\right) + \left(\frac{\frac{1}{2}\lambda_A}{\lambda_A + 2\lambda_B}\right)\left(\frac{\lambda_B}{\frac{1}{2}\lambda_A + \lambda_B}\right) \qquad (42)$$

Which is the final results as the equation (41).

IV. NUMERICAL EXAMPLE

A. THE FUNDAMENTAL DUEL

Two duellists, A and B, start with unloaded weapons and then fire at each other until one is killed. A's firing time (the time between rounds = r_A) is 5 rounds per minute. B's firing time (r_B) is also 5 rounds per minute. Each time A fires, he has a fixed probability $p_A = 0.6$ of killing B. We will denote the probability that B is not killed as $q_A = 0.4$, and hence $p_A + p_B = 1$. Similarly denoted as $p_B = 0.6$, with its complement being similarly defined (i.e., $p_B + q_B = 1$). From the above data, the probability that A's system will win is the following:

$$P(A) = \frac{p_A \, r_A}{p_A r_A + p_B r_B}$$

$$= \frac{0.6X5}{0.6X5 + 0.6X5}$$

$$= 0.5$$

But A's winning chances can be enhanced as his rate of fire and/or kill probability (p_A) increases. From the equation (15),

$$r_B p_B = r_A p_A \left[\frac{1}{P(A)} - 1\right] \tag{43}$$

The following graphs represent the various cases.

CASE 1: $r_A = r_B$

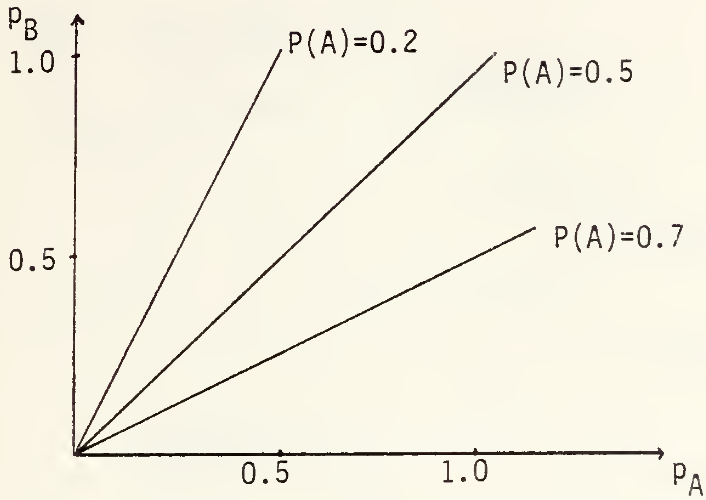

Figure 4. The Relationship of p_A and p_B When $r_A = r_B$

CASE 2: $r_A = 2r_B$

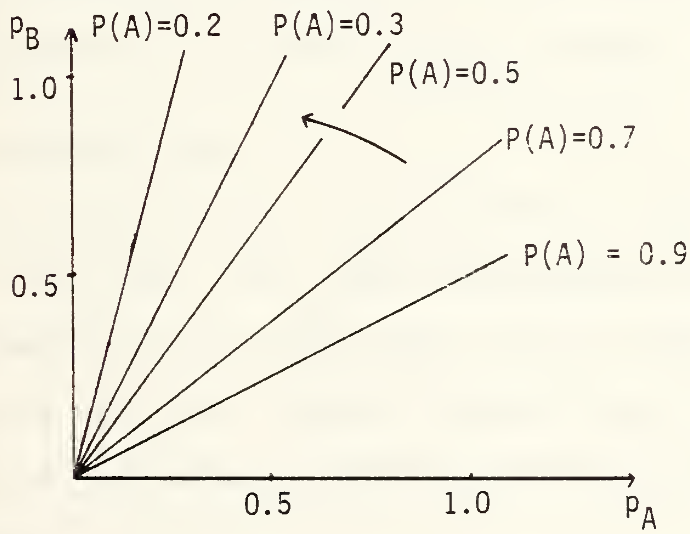

Figure 5. The Relationship of p_A and p_B When $r_A = 2r_B$

If A's rate of fire (r_A) is increased ($r_A = 2r_B$), the contour are rotated count clockwise around the origin.

CASE 3: $r_B p_B = r_A p_A \left[\frac{1}{P(A)} - 1\right]$

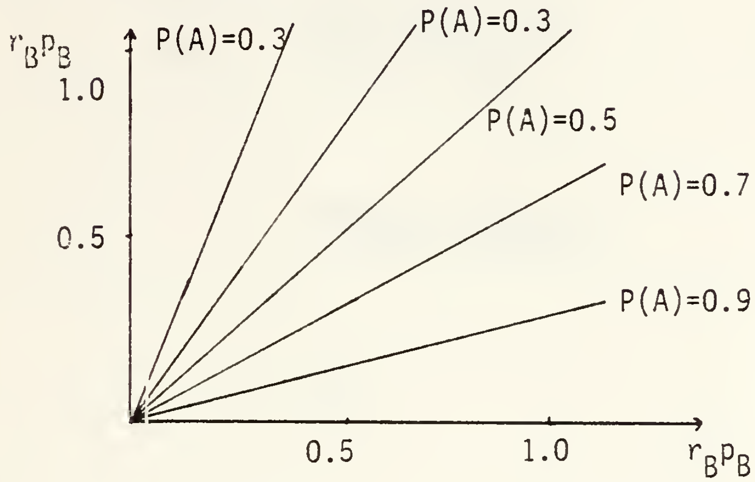

Figure 6. The Relationship of $r_A p_A$ and $r_B p_B$

From Figure 6, A's winning chances (p(A)) are enhanced as his rate of fire (r_A) and/or kill probability (p_A) increases.

B. THE CLASSICAL DUEL

In the classical duel, two duellists, A and B, start with loaded weapons, fire their first rounds simultaneously, and then proceed as in the fundamental duel. Each time A fires, he has a fixed probability $p_A = 0.6$ of killing B. Similarly denoted as $p_B = 0.6$ of killing A. A's firing time is 5 rounds per minutes and B's firing time is also 5 rounds per minute.

Therefore, P(A) can be expressed: $P(A) = p_A q_B + q_A q_B (P_f(A))$ by the equation (16) where $P_f(A)$ is the result of the fundamental duel. By the equation (21),

$$P(A) = \frac{p_A q_B (p_B r_B + r_A)}{p_A r_A + p_B r_B}$$

$$= \frac{0.6X0.4 (0.6X5 + 5)}{0.6X5 + 0.6X5}$$

$$= 0.32$$

Similarly,

$$P(B) = 0.32 \quad .$$

and the probability that both are killed on the first round:

$$P(AB) = 1 - P(A) - P(B) \quad \text{or } P(AB) = p_A p_B \quad .$$

$$= 0.36$$

where P(AB) is the probability that both are killed in the first round.

C. AN EXTENSION TO MULTIPLE FIRES

First, we will consider fundamental duel case when firing time is discrete. In a discrete firing time, two duellists, A and B, start with unlimited ammunition, fire at each other with fixed kill probabilities p_A = 0.6 of killing B. Similarly denoted as p_B = 0.6 of killing A. They start with unloaded weapons and fire at fixed interval a and b respectively. Let's consider a various case of a and b.

1. <u>a = b = 1</u>

From the equation (23) $\quad \frac{\alpha}{\beta} = 1,\ n = 1,\ r = 0$

therefore,

$$P\left[\begin{matrix}\text{A's total success}\\ \text{on the jth round}\end{matrix}\right] = \left\{\frac{p_A q_B^{\,n}}{1 - q_A^{\,\beta} q_B^{\,\alpha}}\right\} \sum_{j=0}^{\beta-1} q_A^{\,j} q_B^{\,jn+[x_j]}$$

$$= 0.285$$

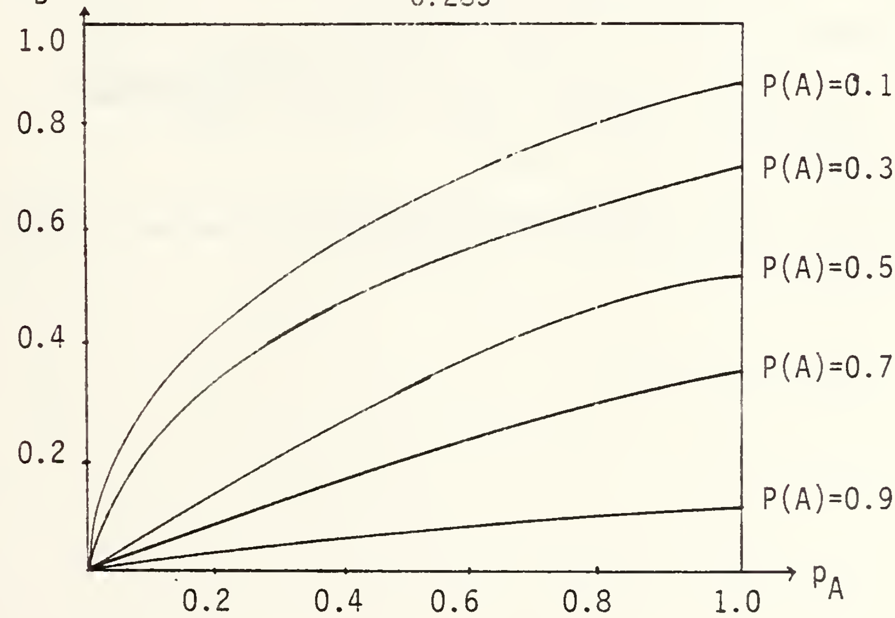

Figure 7. The Relationship Between p_A and p_B When a = b

2. a = 10, b = 5

From the equation (23) $\frac{\alpha}{\beta} = 2,\ n = 2,\ r = 0$

similarly,

$$P\left[\begin{array}{l}\text{A's total success}\\ \text{on the jth round}\end{array}\right] = 0.1$$

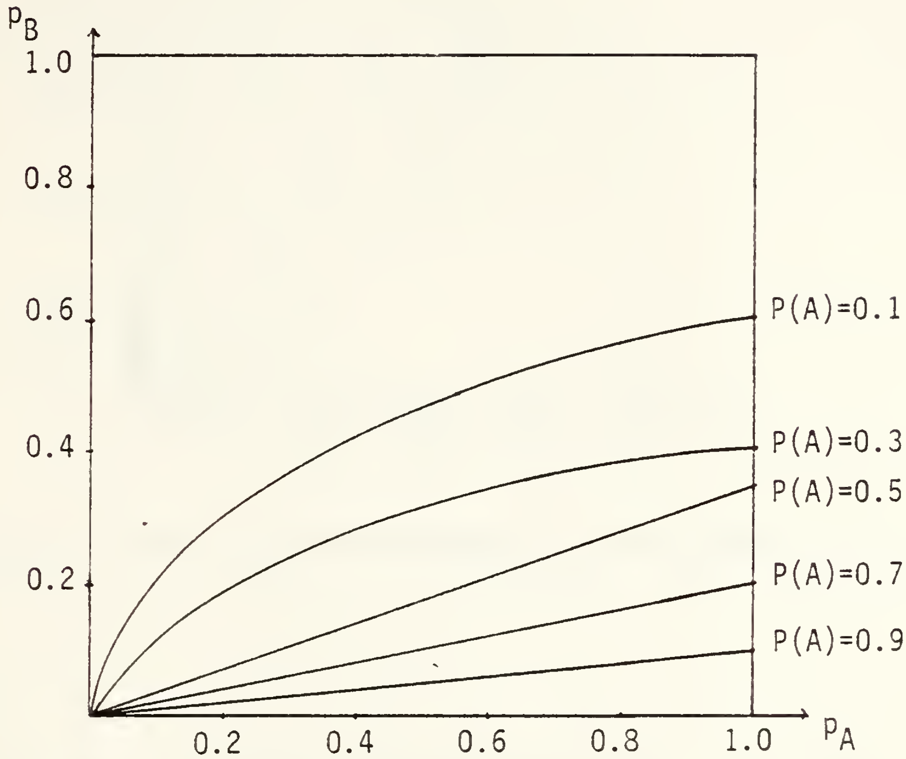

Figure 8. The Relationship Between p_A and p_B When a = 2b

3. <u>a = 5, b = 5</u>

Similarly, $P\left[\begin{array}{l}\text{A's total success}\\ \text{on the jth rounds}\end{array}\right] = 0.743$

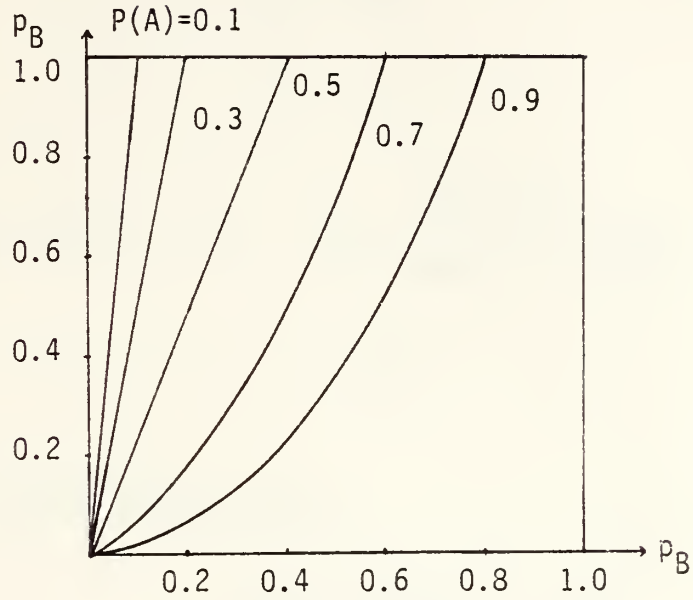

Figure 9. The Relationship Between p_A and p_B When a = ½b

Secondly, we will consider multiple-duel when firing time is discrete. In this duel, there are two combatants on the A's side and one combatant on the B's side as in Figure 1. A (A_1, A_2) has a fixed probability p_A = 0.6 of killing B. Similarly denoted is p_B = 0.6 of killing A. From the mentioned data, we can get the probability that A's system will win. From the equations (32 and (33),

$$P\,[\text{ The side "A" win }] = \frac{p_A\,(1 + q_A p_B - q_A^2 q_B^2)}{(1 - q_A q_B)\,(1 - q_A^2 q_B)}$$

$$= 0.93$$

and

$$P\,[\text{ The side "B" wins }] = \frac{p_B^2\, q_A^3}{(1 - q_A q_B)\,(1 - q_A^2 q_B)}$$

$$= 0.026$$

Similarly, from the equation (38)

$$P\left[\begin{array}{l}\text{Draw of both}\\ \text{sides (AB)}\end{array}\right] = 0.044$$

therefore $\qquad P(A) + P(B) + P(AB) = 1$.

Finally we will consider multiple-duel when firing time is continuous. A's firing time is a random variable with a known probability density, $f_A(t)$. The time between rounds fired is random variable having exponential distribution with $r_A = 5$ round per minute for "A", $r_B = 5$ rounds per minute for "B". The kill probability of "A" sides is $p_A = 0.6$, and $p_B = 0.6$. Therefore, from equations (41) and (42) we can get P(A) and P(B):

$$P(A) = P_2 \cdot P_6 + P_1 \cdot P_7$$

$$= \frac{(\tfrac{1}{2}\lambda_A)\,(\tfrac{1}{2}\lambda_A)}{(\lambda_A + 2\lambda_B)\cdot(\lambda_B + \tfrac{1}{2}\lambda_A)} + \frac{(\tfrac{1}{2}\lambda_A)\,(\tfrac{1}{2}\lambda_A)}{(\lambda_A + 2\lambda_B)\,(\tfrac{1}{2}\lambda_A + \lambda_B)}$$

$$= 0.11$$

and similarly,

$$P(B) = P_3 + P_2 \cdot P_5 + P_1 \cdot P_8$$

$$= \frac{2\lambda_B}{\lambda_A + 2\lambda_B} + \left(\frac{\tfrac{1}{2}\lambda_A}{\lambda_A + 2\lambda_B}\right)\left(\frac{\lambda_B}{\lambda_B + \tfrac{1}{2}\lambda_A}\right) + \left(\frac{\tfrac{1}{2}\lambda_A}{\lambda_A + 2\lambda_B}\right)\left(\frac{\lambda_B}{\tfrac{1}{2}\lambda_A + \lambda_B}\right)$$

$$= 0.89$$

where $\lambda_A = r_A p_A$ and $\lambda_B = r_B p_B$.

V. SUGGESTED FUTURE WORK

Models investigated in this paper include simple stochastic models and a multiple duel model using the theory of continuous-time Markov chains. The standard case was unlimited time, unlimited ammunition, and a fixed kill probability. Models in which both time and ammunition are limited would be desirable. Numerous extensions and modifications of the fundamental-duel can be further studied as follows [Ref. 4]:

CASE 1: One-Versus-One

(1) Variable Kill Probability - p_A and p_B are special functions of time and round dependent kill probability.

(2) Duel with initial suprise - random initial suprise

(3) Fixed ammunition supply, etc.

CASE 2: Two-Versus-Two

(1) Several multiple:

$$\left\{\begin{matrix} A \searrow & \longleftarrow B \\ A \longrightarrow & \longleftarrow B \end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix} A \longrightarrow & \longleftarrow B \\ A \longrightarrow & \longleftarrow B \end{matrix}\right\}$$

where A and B are contestants.

(2) Round dependent kill probability, connection with Lanchester's models.

However, these suggested models with more than two contestants may be limited to simple situations because the uncoupling principle which is used to solve the fundamental-duel is no longer applicable.

Consequently, we must consider each event as it occurs, as well as all the possible interactions and conditional events that may occur subsequently.

VI. FINAL REMARKS

Simple stochastic models for the fundamental-duel and the classical-duel have been reviewed and analyzed by the graphical methods. For the extension to multiple-duels two situations have been considered: 1) discrete firing times, and 2) continuous firing times. When the firing time is discrete, we are able to examine some duels in which strong interactions occur by limiting our consideration to those situations in which the time between rounds is constant. When the firing time is continuous random variables, an expression for the probability of winning such a duel is derived by using the theory of continuous-time Markov chains. Numerical examples for each model are presented. Still there is much work left to be done in the future.

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